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SOME DYNAMIC PROBLEMS OF THE THEORY OF ELASTICITY

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On the basis of the functionally-invariant solutions of the wave equation, suggested by Smirnov and Sobolev, we give a closed solution of a class of selfsimilar problems of the dynamical theory of elasticity. This class contains the following problems: (a) a half-plane, arbitrarily loaded at the boundary (including the case when the endpoints of the loaded segments move with arbitrary constant velocities); (b) the contact problem for the half-plane, when the ends of the contact areas are displaced with arbitrary constant velocities; (c) a collection of arbitrarily loaded cuts along the same line, moving with constant velocities, the different endpoints of the cuts having, possibly, different velocities. The solution of the indicated problems are reduced in the simplest cases to the Dirichlet problem or to the mixed Keldysh-Sedov problems of the theory of analytic functions of a complex variable. In principle, the procedure for finding the solution

is not more complicated than for the similar problems of statics and steady dynamics (the solution of the latter problems has been found basically by Kolosov, Muskhelishvili, Galin and Radok). First we introduce the general representations of the solutions by analytic functions of the complex variable for an arbitrary index of selfsimilarity and we describe the general method of solution. Then we illustrate the method with some concrete problems of the indicated class. The examination is restricted to the plane problems for the homogeneous and isotropic body; however the method can be easily generalized to the case of an anisotropic piecewise-homogeneous body, when the upper and lower half-planes have different elastic constants.

In 1932, Smirnov and Sobolev have discovered a class of solutions of the wave equation, in which the solution is represented in terms of an analytic function of a complex variable [1] (see the Appendix). In particular, this class contains the selfsimilar problems. Some interesting problems from the theory of the diffraction of plane elastic waves at a cut [2, 3] and at a wedge have been analyzed by the Smirnov-Sobolev method, however, in the majority of the solutions of the dynamical problems of the theory of elasticity, obtained in the last 20 years, considerably more cumbersome methods have been used, requiring a large volume of computational work. We enumerate some of the most well-known solutions.

The problem of the sudden appearance of a rectilinear semi-infinite fixed cut in a constant tension field has been solved [4] and also the problem in which the same cut moves with a fixed velocity from the instant of its appearance [5]. The propagation of a crack with a constant velocity in both directions, with initial length equal to zero in a constant tension field has been investigated [6] (the corresponding axisymmetric problem has been considered in [7, 8]. Some dynamical problems of crack propagation are studied in [12] and a sufficiently detailed survey of the literature is given there. In [13 – 15] Broberg's problem [6] has been generalized to the case of an anisotropic material and to the case of an arbitrarily given normal load on the crack which maintains the selfsimilar character of the problem. All the indicated solutions are selfsimilar with index (0, 0) (see monograph [1] and also Sect. 1 of the present paper).

To these problems belong also the contact problem of the impact of a wedge on a half-plane [18], the problem of the motion with a constant velocity of a concentrated force on the boundary of a half-plane starting at some initial instant [19], the contact problem on the thrust of a die against a half-plane [20]. In the problems [18 – 20] the index of selfsimilarity is different. A large number of analogous selfsimilar problems has been considered in the acoustic approximation which correspond to shear in the theory of elasticity (see, for example, [21]).

In recent papers [22, 23], simultaneously and independently, the problem of the motion of a semiinfinite cut with a constant velocity (as in Baker's problem) has been solved; concentrated forces are applied to the sides of the cut. This solution can be used as the Green's function in the case of arbitrary static loads. Making use of the characteristic property of the stress intensity coefficient in the obtained solution, one has succeeded to generalize it to the case of the motion of the cut with an arbitrary nonconstant velocity for arbitrary applied loads [22].

In the present paper we develop a general approach to the selfsimilar problems of the dynamical theory of elasticity, which allows us to obtain their solution in a very simple way in closed form. This approach is based on the general representations of the solutions in terms of analytic functions of a complex variable which allows us to formulate at once the indicated selfsimilar problems as some Riemann-Hilbert problems for the half-plane (in the simplest cases one obtains the Dirichlet problem and the mixed Keldysh-Sedov problem). The Riemann-Hilbert problem for the half-plane can be easily solved with the aid of the standard methods, presented for example, in [24, 25]. In Sects. 2 – 4 we consider also some concrete problems of punches, of moving loads and of cracks. The indicated general approach has been applied previously to the solution of different particular problems [26 – 30].

1. General representations. The method of solution. We assume that a homogeneous and isotropic elastic body is in a state of plane strain. The fundamental equations of the dynamic theory of elasticity have in this case the following form:

$$u = u_1 + u_2, \qquad v = v_1 + v_2 \tag{1.1}$$

$$\frac{\partial u_1}{\partial y} = \frac{\partial v_1}{\partial x}, \qquad \frac{\partial u_2}{\partial x} = -\frac{\partial v_2}{\partial y}$$

$$\Delta u_k = \frac{1}{c_k^2}\frac{\partial^2 u_k}{\partial t^2}, \quad \Delta v_k = \frac{1}{c_k^2}\frac{\partial^2 v_k}{\partial t^2} \quad (k = 1, 2) \cdot \tag{1.2}$$

$$\left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

Here $u\ (x, y, t)$ and $v\ (x, y, t)$ are the displacement components with respect to the axes x and y of the Cartesian coordinates, respectively, c_1 and c_2 are the velocities of the longitudinal and transverse waves $(c_1 > c_2)$.

The components σ_x, σ_y, τ_{xy} of the stress tensor are, according to Hooke's law (μ is Lamé's constant)

$$\begin{aligned}
\sigma_x &= \mu\left[\frac{c_1^2}{c_2^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - 2\frac{\partial v}{\partial y}\right] \\
\sigma_y &= \mu\left[\frac{c_1^2}{c_2^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - 2\frac{\partial u}{\partial x}\right] \\
\tau_{xy} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)
\end{aligned} \tag{1.3}$$

We indicate a class of selfsimilar plane problems of the dynamic theory of elasticity, whose solution with the aid of the complex variables z_1 and z_2, where

$$z_k = \frac{xt - iy\sqrt{t^2 - c_k^{-2}(x^2 + y^2)}}{x^2 + y^2} \quad (k = 1, 2) \tag{1.4}$$

reduces to the Riemann-Hilbert boundary value problem from the theory of analytic functions of one complex variable (in the simplest cases to the Dirichlet problem or to the mixed Keldysh-Sedov problem). This class includes the following problems [17].

The first fundamental problem (problem A): (a) an infinite elastic half-space $y \geqslant 0$ has an arbitrary number of loaded segments along the x-axis, the endpoints of these segments, having coordinates x_n, are moving with constant velocities V_n, so that $x_n = V_n t$ (in particular, the velocities of some endpoints may be equal to zero):

b) at the initial instant $t = 0$ the half-space is at rest;

c) the normal and tangential loads on the indicated segments are arbitrary linear combinations of the following functions:

$$\frac{d^m f_n(x)}{dx^m} \frac{d^k f_l(t)}{dt^k} \tag{1.5}$$

where

$$f_i(\xi) = \begin{cases} 0 & \text{for } \xi < 0 \\ \xi^i & \text{for } \xi > 0 \end{cases} \tag{1.6}$$

Here k, l, m, n are arbitrary positive integers.

The second fundamental problem (problem B): (a) the infinite elastic space has an arbitrary number of cuts along the x-axis, the endpoints x_n of the cuts moving with constant velocities V_n, so that $x_n = V_n t$ (in particular, some V_n may be equal to zero);

b) at the initial instant $t = 0$ the space is at rest;

c) the normal and tangential loads on the cuts are arbitrary linear combinations of the functions (1.5) (see also (1.6)), the loaded segments need not coincide with the cuts and may move with their own velocities.

The contact problem (problem C): (a) an infinite elastic semispace $y \geqslant 0$ has an arbitrary number of loaded segments along the x-axis, the endpoints of these segments with coordinates x_n, are moving with constant velocities, so that $x_n = V_n t$;

b) at the initial instant $t = 0$ the space is at rest;

c) on the loaded segments we have boundary conditions of one of the three types: (1) the normal and tangential displacements are given as some arbitrary linear combinations of functions of the form (1.5), (1.6) (a rough punch); (2) the tangential stresses are equal to zero and the normal displacements are linear combinations of functions of the form (1.5), (1.6) (a smooth punch); (3) the tangential stress is directly proportional to the normal stress (i.e. Coulomb's law $\tau_{xy} = k\sigma_y$) of dry friction is given). and the normal displacements are linear combinations of functions of the form (1.5), (1.6).

For $m > n$, the first factor in (1.5) represents, except for a numerical coefficient, the $(m - n + 1)$-th derivative of the Dirac delta function; this remark refers also to the second factor, with the obvious correspondence of the indices $m \to k$, $n \to l$.

The type of the selfsimilar Problem A, B or C is determined by the two numbers $m - n$ and $k - l$. The pair of numbers $(m - n, k - l)$ will be called the selfsimilarity index of the Problems A, B, C. In order to solve a problem of the above mentioned type, one has first to determine the selfsimilarity index of the boundary value problem, or else, to represent the boundary conditions in the form of a linear superposition of selfsimilar problems with different indices. Then, for each index one has to introduce the corresponding general representations in terms of analytic functions and then to solve the concrete boundary value problems of the theory of analytic functions obtained from the boundary conditions, In the general case of the boundary conditions given above, it is necessary to solve the Riemann-Hilbert boundary value problem with discontinuous coefficients for one complex variable in the halfplane; this is a well studied problem and its closed solution can be obtained without difficulty (see, for example, [24, 25]).

We note at once the following two circumstances.

1). An arbitrary continuous function of two variables x and t in any closed domain can be uniformly approximated by some polynomial, i.e. a sum of terms of the form $x^m t^n$. To each of these terms in the boundary conditions of the Problems A, B and C (see (1.5) and (1.6)) there corresponds a well defined selfsimilarity index. Consequently, the case when the given loads or displacements represent arbitrary continuous functions reduces to the considered case. In addition, an arbitrary function of x and t can be represented in the form of a linear superposition of δ_0-shaped and δ_1-shaped functions, for each of which the solution will be selfsimilar and can be used as a Green's function.

2). From the solutions of the boundary value Problems A, B and C, the solution of the corresponding static and steady dynamic problems of the plane theory of elasticity can be obtained as some limiting cases. We indicate the corresponding limiting processes:

The limiting quasi-static problem

$$V_n \to 0, \qquad t \to \infty, \qquad V_n t \to a_n \tag{1.7}$$

the limiting quasi-steady dynamic problem

$$V_n = V + \varepsilon_n, \quad \varepsilon_n \to 0, \quad t \to \infty, \quad \varepsilon_n t \to a_n \tag{1.8}$$

Here V and a_n are some constants.

Consequently, the indicated class of solutions of the dynamic theory of elasticity is, essentially, the analog of the plane static problem for the half-plane [31] and of the plane steady dynamic problem for the half-plane [32, 33].

We deduce general representations for the most commonly used selfsimilar types. From dimensional analysis it follows that for every selfsimilarity index there exist functions which satisfy the wave equation and are homogeneous functions of x, y, t of dimension zero. These are the following functions:

1° the displacements u and v, Problem C, index (1, 0);

2° the stresses σ_x, σ_y, τ_{xy}, Problems A and B, index (0, 0);

3° the displacement potentials φ and ψ, Problems A and B, index (1, 1); we recall that

$$u = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x} \tag{1.9}$$

4° the functions Lu and Lv, Problem C, index (m, n);

5° the functions $L\sigma_x$, $L\sigma_y$, $L\tau_{xy}$, Problems A and B, index (m, n).

In 4° and 5°, by L we understand the following linear differential operator:

$$L = \frac{\partial^{m+n}}{\partial x^m \, \partial t^n} \tag{1.10}$$

We represent the indicated homogeneous functions of x, y, t of dimension zero, in the form of a sum of two terms; one of them, obviously, satisfies the wave equation for longitudinal waves, and the other one satisfies the wave equation for transverse waves. The first of these terms can be represented as the real part of some analytic function of z_1, while the second one as the real part of another analytic function of z_2 (see the Appendix and formula (1.11)). For $y = 0$ we have

$$z_1 = z_2 = t / x \qquad (y=0) \tag{1.11}$$

Introducing the complex variable z, whose real part is equal to t / x, we reduce the boundary value Problems A, B or C of one index to the Riemann-Hilbert boundary value problem of one complex variable z but for several functions. However, in the case under consideration, all the functions can be expressed in terms of one function, and the problem reduces to the standard Riemann-Hilbert problem for one function. In the simplest cases we obtain the Dirichlet problem and the mixed Keldysh-Sedov problem.

Let us find the general representation for the five cases indicated.

1°. The displacements u and v are homogeneous functions. According to the formulas (1.2) and (1.4), we have

$$u = \operatorname{Re}[f_1(z_1) + f_2(z_2)], \qquad v = \operatorname{Re}[f_3(z_1) + f_4(z_2)] \tag{1.12}$$

The four unknown functions of complex variables must satisfy the two conditions (1.1) which take the following form:

$$\begin{aligned} \operatorname{Re}\left[f_1'(z_1)\frac{\partial z_1}{\partial y}\right] &= \operatorname{Re}\left[f_3'(z_1)\frac{\partial z_1}{\partial x}\right] \\ \operatorname{Re}\left[f_2(z_2)\frac{\partial z_2}{\partial x}\right] &= -\operatorname{Re}\left[f_4'(z_2)\frac{\partial z_2}{\partial y}\right] \end{aligned} \tag{1.13}$$

Differentiating (A. 7), we find

$$\frac{\partial z_k}{\partial x} = \frac{z_k\sqrt{c_k^{-2} - z_k^2}}{yz_k - x\sqrt{c_k^{-2} - z_k^2}}, \quad \frac{\partial z_k}{\partial y} = \frac{c_k^{-2} - z_k^2}{yz_k - x\sqrt{c_k^{-2} - z_k^2}} \quad (k = 1, 2) \tag{1.14}$$

Hence

$$\begin{aligned} \sqrt{c_1^{-2} - z_1^2}\, f_1'(z_1) &= z_1 f_3'(z_1) \\ z_2 f_2'(z_2) &= -\sqrt{c_2^{-2} - z_2^2}\, f_4'(z_2) \end{aligned} \tag{1.15}$$

Thus, the homogeneous displacements u and v can be represented with the aid of (1.12) in terms of four analytic functions related by the two equalities (1.15). The representations for the stresses are obtained from (1.12), (1.14), (1.15) by making use of (1.3)

$$\begin{aligned} \sigma_x &= \mu \operatorname{Re}\left[\frac{c_2^{-2} - 2(c_1^{-2} - z_1^2)}{z_1^2} f_1'(z_1)\frac{\partial z_1}{\partial x} + 2f_2'(z_2)\frac{\partial z_2}{\partial x}\right] \\ \sigma_y &= \mu \operatorname{Re}\left[\frac{c_2^{-2} - 2z_1^2}{z_1^2} f_1'(z_1)\frac{\partial z_1}{\partial x} - 2f_2'(z_2)\frac{\partial z_2}{\partial x}\right] \\ \tau_{xy} &= \mu \operatorname{Re}\left[2f_1'(z_1)\frac{\partial z_1}{\partial y} + \frac{c_2^{-2} - 2z^2}{z_2\sqrt{c_2^{-2} - z_2^2}} f_2'(z_2)\frac{\partial z_2}{\partial x}\right] \end{aligned} \tag{1.16}$$

2°. The stresses σ_x, σ_y, τ_{xy} are homogeneous functions. According to Hooke's law, the first derivatives of the displacements with respect to the coordinates are also homogeneous functions. Consequently, they can be represented in the form

$$\begin{aligned} \frac{\partial u}{\partial x} &= \operatorname{Re}[f_1(z_1) + f_2(z_2)], & \frac{\partial u}{\partial y} &= \operatorname{Re}[f_3(z_1) + f_4(z_2)] \\ \frac{\partial v}{\partial x} &= \operatorname{Re}[f_5(z_1) + f_6(z_2)], & \frac{\partial v}{\partial y} &= \operatorname{Re}[f_7(z_1) + f_8(z_2)] \end{aligned} \tag{1.17}$$

The eight unknown functions of complex variables in (1.17) must satisfy the following six conditions, four of which are consequences of (1.1):

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right), \qquad \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) \tag{1.18}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial u_1}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial v_1}{\partial x}\right), \qquad \frac{\partial}{\partial y}\left(\frac{\partial u_1}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial v_1}{\partial y}\right)$$
$$\frac{\partial}{\partial x}\left(\frac{\partial u_2}{\partial x}\right)=-\frac{\partial}{\partial y}\left(\frac{\partial v_2}{\partial x}\right), \qquad \frac{\partial}{\partial x}\left(\frac{\partial u_2}{\partial y}\right)=-\frac{\partial}{\partial y}\left(\frac{\partial v_2}{\partial y}\right)$$

Finally, making use of the formulas (1.14), we find the representations

$$\frac{\partial u}{\partial x}=\mathrm{Re}\,[g_1(z_1)+g_2(z_2)], \qquad \frac{\partial u}{\partial y}=\mathrm{Re}\,[g_3(z_1)+g_4(z_2)] \tag{1.19}$$
$$\frac{\partial v}{\partial x}=\mathrm{Re}\,[g_3(z_1)+g_5(z_2)], \qquad \frac{\partial v}{\partial y}=\mathrm{Re}\,[g_6(z_1)-g_2(z_2)]$$

Here the six unknown functions of a complex variable must satisfy the following four equalities resulted from (1.18):

$$z_1 g_3'(z_1)=\sqrt{c_1^{-2}-z_1^2}\,g_1'(z_1), \quad z_1 g_6'(z_1)=\sqrt{c_1^{-2}-z_1^2}\,g_3'(z_1) \tag{1.20}$$
$$z_2 g_2'(z_2)=-\sqrt{c_2^{-2}-z_2^2}\,g_5'(z_2), \quad z_2 g_4'(z_2)=-\sqrt{c_2^{-2}-z_2^2}\,g_2'(z_2)$$

Substituting (1.19) into the formulas (1.3), we obtain the stresses

$$\sigma_x=\mu\,\mathrm{Re}\left\{\frac{c_1^2}{c_2^2}\,[g_1(z_1)+g_6(z_1)]-2g_6(z_1)+2g_2(z_2)\right\} \tag{1.21}$$
$$\sigma_y=\mu\,\mathrm{Re}\left\{\frac{c_1^2}{c_2^2}\,[g_1(z_1)+g_6(z_1)]-2g_1(z_1)-2g_2(z_2)\right\}$$
$$\tau_{xy}=\mu\,\mathrm{Re}\,\{2g_3(z_1)+g_4(z_2)+g_5(z_2)\}$$

3°. The potentials φ and ψ are homogeneous functions. Since φ satisfies the wave equation for longitudinal waves and ψ for transverse waves, the functions φ and ψ can be represented in the form

$$\varphi=\mathrm{Re}\,f_1(z_1), \qquad \psi=\mathrm{Re}\,f_2(z_2) \tag{1.22}$$

Substituting (1.21) into (1.3) and (1.9), we have

$$u=\mathrm{Re}\left[f_1'(z_1)\frac{\partial z_1}{\partial x}+f_2'(z_2)\frac{\partial z_2}{\partial y}\right] \tag{1.23}$$
$$v=\mathrm{Re}\left[f_1'(z_1)\frac{\partial z_1}{\partial y}-f_2'(z_2)\frac{\partial z_2}{\partial x}\right]$$

$$\sigma_x=-\mu\,\mathrm{Re}\,\frac{\partial}{\partial t}\left[\frac{c_2^{-2}-2(c_1^{-2}-z_1^2)}{z_1}f_1'(z_1)\frac{\partial z_1}{\partial x}+2z_2 f_2'(z_2)\frac{\partial z_2}{\partial y}\right] \tag{1.24}$$
$$\sigma_y=-\mu\,\mathrm{Re}\,\frac{\partial}{\partial t}\left[\frac{c_2^{-2}-2z_1^2}{z_1}f_1'(z_1)\frac{\partial z_1}{\partial x}-2z_2 f_2'(z_2)\frac{\partial z_2}{\partial y}\right]$$
$$\tau_{xy}=-\mu\,\mathrm{Re}\,\frac{\partial}{\partial t}\left[2z_1 f_1'(z_1)\frac{\partial z_1}{\partial y}+\frac{c_2^{-2}-2z_2^2}{z_2}f_2'(z_2)\frac{\partial z_2}{\partial x}\right]$$

4°. The functions Lu, Lv are homogeneous. In this case for every linear differential operator L, we have the representations (1.12), (1.16) where instead of the displacements u, v and the stresses σ_x, σ_y, τ_{xy} we set Lu, Lv and $L\sigma_x$, $L\sigma_y$, $L\tau_{xy}$, respectively.

5°. The functions $L\sigma_x, L\sigma_y, L\tau_{xy}$ are homogeneous. In this case the representations (1.19) − (1.21) are valid, where instead of the displacements u, v and the stresses $\sigma_x, \sigma_y, \tau_{xy}$ we set Lu, Lv and $L\sigma_x, L\sigma_y, L\tau_{xy}$ respectively.

Thus, in all the indicated representations, essentially only two independent unknown functions of a complex variable are involved, as in the static problems of the theory of elasticity.

The general representations of the problems, symmetric with respect to the x-axis. We present a sufficiently large class of problems of the dynamical theory of elasticity (symmetric with respect to the x-axis) for whose solutions we pose the following boundary condition:

$$\tau_{xy} = 0 \quad \text{for} \quad y = 0 \tag{1.25}$$

In this case we can obtain the general representation of the solution in terms of one analytic function (of the same form for all the selfsimilar types indicated above).

The boundary condition (1.25), on the basis of the representations (1.12) and (1.16), (1.17) and (1.21), (1.22) − (1.23) leads to the following supplementary equalities:

homogeneous displacements

$$2\sqrt{c_1^{-2} - z^2}\sqrt{c_2^{-2} - z^2}\, f_1'(z) + (c_2^{-2} - 2z^2) f_2'(z) = 0 \tag{1.26}$$

homogeneous stresses

$$2g_3(z) + g_4(z) + g_5(z) = 0 \tag{1.27}$$

homogeneous potentials

$$2z\sqrt{c_1^{-2} - z^2} f'_1(z) + (c_2^{-2} - 2z^2) f_2'(z) = 0 \tag{1.28}$$

These equalities together with the equalities (1.15), (1.20) or (1.22) (respectively) allow us to espress all the quantities in terms of only one unknown analytic function of a complex variable.

A similar class of problems in the statical theory of elasticity has been found by Westergaard [34].

We introduce the following notation:

in the case when the functions Lu, Lv are homogeneous

$$u^\circ = Lu, \qquad v^\circ = Lv \tag{1.29}$$
$$\sigma_x^\circ = L\sigma_x, \qquad \sigma_y^\circ = L\sigma_y, \qquad \tau_{xy}^\circ = L\tau_{xy}$$

in the case when the functions $L\sigma_x$, $L\sigma_y$, $L\tau_{xy}$ are homogeneous

$$u^\circ = \frac{\partial}{\partial t} Lu, \qquad v^\circ = \frac{\partial}{\partial t} Lv \tag{1.30}$$
$$\sigma_x^\circ = \frac{\partial}{\partial t} L\sigma_x, \quad \sigma_y^\bullet = \frac{\partial}{\partial t} L\sigma_y, \quad \tau^\circ{}_y = \frac{\partial}{\partial t} L\tau_{xy}$$

in the case when the potentials φ, ψ are homogeneous

$$u^\circ = \int_0^t u(x, y, \tau)\, d\tau, \quad v^\circ = \int_0^t v(x, y, \tau)\, d\tau \tag{1.31}$$
$$\sigma_x^\circ = \int_0^t \sigma_x(x, y, \tau)\, d\tau, \quad \sigma_y^\circ = \int_0^t \sigma_y(x, y, \tau)\, d\tau, \quad \tau_{xy} = \int_0^t \tau_{xy}(x, y, \tau)\, d\tau$$

In all the indicated cases the functions u° and v° are homogeneous. With the aid of the notation introduced, all the general representations in terms of one analytic function can be reduced to the following form:

$$u^\circ = \operatorname{Re}[U_1(z_1) + U_2(z_2)], \quad v^\circ = \operatorname{Re}[V_1(z_1) + V_2(z_2)] \tag{1.32}$$

$$\sigma_x^\circ = \frac{\mu}{c_2^{-2}} \operatorname{Re}\left\{\frac{[c_2^{-2} - 2(c_1^{-2} - z_1^2)](c_2^{-2} - 2z_1^2)}{c_1^{-2} - z_1^2} W'(z_1)\frac{\partial z_1}{\partial y} - 4z_2^2 W'(z_2)\frac{\partial z_2}{\partial y}\right\} \tag{1.33}$$

$$\sigma_y^\circ = \frac{\mu}{c_2^{-2}} \operatorname{Re}\left\{\frac{(c_2^{-2} - 2z_1^2)^2}{c_1^{-2} - z_1^2} W'(z_1)\frac{\partial z_1}{\partial y} + 4z_2^2 W'(z_2)\frac{\partial z_2}{\partial y}\right\}$$

$$\tau_{xy} = \frac{\mu}{c_2^{-2}} \operatorname{Re}\left\{2(c_2^{-2} - 2z_1^2) W'(z_1)\frac{\partial z_1}{\partial x} - 2(c_2^{-2} - 2z_2^2) W'(z_2)\frac{\partial z_2}{\partial x}\right\}$$

Here the functions $U_k(z)$ and $V_k(z)$ are expressed in terms of the function $W(z)$ as:

$$U_1'(z) = \frac{z(c_2^{-2} - 2z^2)}{c_2^{-2}\sqrt{c_1^{-2} - z^2}} W'(z), \qquad V_1'(z) = \frac{c_2^{-2} - 2z^2}{c_2^{-2}} W'(z) \tag{1.34}$$

$$U_2'(z) = -\frac{2z\sqrt{c_2^{-2} - z^2}}{c_2^{-2}} W'(z), \qquad V_2'(z) = \frac{2z^2}{c_2^{-2}} W'(z)$$

The functions $v^\circ(x, 0, t)$ and $\sigma_y^\circ(x, 0, t)$ are expressed in terms of the function $W(z)$ in the following manner:

$$y = 0, \quad v^\circ = \operatorname{Re} W(z) \quad (z = t/x) \tag{1.35}$$

$$y = 0, \quad \sigma_y^\circ = \frac{1}{t}\operatorname{Re}\left\{-\frac{\mu}{c_2^{-2}} \frac{zS(z)}{\sqrt{c_1^{-2} - z^2}} W'(z)\right\} \tag{1.36}$$

Here and in the sequel, $S(z)$ denotes the function

$$S(z) = (c_2^{-2} - 2z^2)^2 + 4z^2\sqrt{c_1^{-2} - z^2}\sqrt{c_2^{-2} - z^2} \tag{1.37}$$

analytic in the exterior of the cuts $(-c_2^{-1}, -c_1^{-1})$ and (c_1^{-1}, c_2^{-1}) of the plane z and real for $\operatorname{Im} z = 0$ outside these cuts.

We note that the unique real root of the equation $S(z) = 0$ is the quantity which is the reciprocal of the velocity of the Rayleigh wave.

In the problems with given applied loads it is convenient to make use of the function $F(z)$, analytic in the upper (lower) half-plane

$$F(z) = -\frac{\mu}{c_2^{-2}} \frac{zS(z)}{\sqrt{c_1^{-2} - z^2}} W'(z) \tag{1.38}$$

Let us note the following formula which follows from the relations (1.36) and (1.38):

$$y = 0, \quad \sigma_y^\circ = t^{-1}\operatorname{Re} F(z) \quad (z = t/x) \tag{1.39}$$

2. The first fundamental problem. The boundary conditions of the first fundamental problem for the half-plane are the following:

$$\sigma_y = f_1(x, t), \qquad \tau_{xy} = f_2(x, t) \qquad \text{for } y = 0 \tag{2.1}$$

Problem 2.1. The problem on the action of an instantaneous concentrated impulse 1, applied to the boundary of the half-plane, is formulated in the following manner:

$$\sigma_y = -I\delta_1(x)\delta_1(t), \qquad \tau_{xy} = 0 \qquad \text{for } y = 0 \tag{2.2}$$

where $\delta_1(x)$ is the Dirac function. This is Lamb's classical problem.

In this case the potentials φ and ψ are homogeneous and we can make use of the formulas (1.31) – (1.39). With the aid of the formulas (1.39) and (1.31), the boundary conditions (2.2) are written as:

$$\operatorname{Im} z = 0, \qquad \operatorname{Re} F(z) = -I\delta_1\left(\frac{x}{t}\right) \quad \left(z = \frac{t}{x}\right) \tag{2.3}$$

Here we have made use of the relation

$$t\delta_1(x) = \delta_1\left(\frac{x}{t}\right)$$

from the theory of generalized functions. Condition (2.3) can be written

$$\operatorname{Im} z = 0, \qquad \operatorname{Re} F\left(\frac{1}{z}\right) = -I\delta_1(z) \tag{2.4}$$

The solution of the Dirichlet problem (2.4) has the form [24, 25]

$$F\left(\frac{1}{z}\right) = -\frac{Ii}{\pi z}, \qquad F(z) = -\frac{Ii}{\pi} z \tag{2.5}$$

From the formula (1.38) we obtain

$$W'(z) = \frac{iIc_2^{-2}}{\pi\mu} \frac{\sqrt{c_1^{-2} - z^2}}{S(z)} \tag{2.6}$$

Problem 2.2. Assume that a constant concentrated force P for $t \geqslant 0$ is applied perpendicular to the boundary of the half-plane and moves with a constant velocity V along the x-axis, while for $t < 0$ half-plane was at rest. The boundary conditions of this problem are the following:

$$\sigma_y = -P\delta_1(x - Vt)\,\delta_0(t), \qquad \tau_{xy} = 0 \qquad \text{for } y = 0 \tag{2.7}$$

where $\delta_0(t)$ is the Heaviside function. By definition $\delta_0'(t) = \delta_1(t)$. In this case the displacements are, obviously, homogeneous functions. According to the formulas (1.39) and (1.29), in which $L = 1$, the boundary conditions (2.7) can be written in the following form:

$$\operatorname{Im} z = 0, \quad \operatorname{Re} F(z) = -P\delta_1(x - Vt)\, t\delta_0(t) \tag{2.8}$$

$$(z = t/x, \ t > 0)$$

Since for $t > 0$

$$\delta_0(t) = 1, \qquad t\delta_1(x - Vt) = \delta_1(x/t - V)$$

the boundary conditions (2.8) can be written

$$\operatorname{Im} z = 0, \qquad \operatorname{Re} F\left(\frac{1}{z}\right) = -P\delta_1(z - V) \tag{2.9}$$

The solution of the Dirichlet problem (2.9) has the following form:

$$F\left(\frac{1}{z}\right) = -\frac{Pi}{\pi(z - V)}$$

i.e.

$$F(z) = -\frac{iPV^{-1}z}{\pi(V^{-1} - z)} \tag{2.10}$$

According to formula (1.38) we have

$$W'(z) = \frac{Pic_2^{-2}V^{-1}\sqrt{c_1^{-2} - z^2}}{\pi\mu S(z)(V^{-1} - z)} \tag{2.11}$$

The displacements and the stresses are obtained from here with the aid of the formulas (1.29), (1.32) – (1.34) for $L = 1$. For $V = 0$ from (2.11) we obtain the corresponding solution for the fixed force. The paper [19] contains the solution of Problem 2.2 by the method of integral transformations.

Problem 2.3. Let us assume that a constant pressure p is propagating at the boundary of the half-plane $y > 0$ with a constant velocity V in both directions of the x-axis; for $|x| > Vt$ the boundary is free of loads

$$\sigma_y = -p, \quad \tau_{xy} = 0 \quad \text{for} \quad y = 0, \quad |x| < Vt \qquad (2.12)$$
$$\sigma_y = 0, \quad \tau_{xy} = 0 \quad \text{for} \quad y = 0, \quad |x| > Vt$$

In this problem, obviously, the stresses are homogeneous functions. According to the formulas (1.30) and (1.39), where it is necessary to set $L = 1$, the boundary conditions (2.12) take the form

$$\operatorname{Im} z = 0, \quad \operatorname{Re} F(z) = -pVt\,[\delta_1(x - Vt) + \delta_1(x + Vt)] \qquad (2.13)$$
$$(z = t/x, \ t > 0)$$

Since for $t > 0$

$$t\delta_1(x - Vt) = \delta_1\left(\frac{x}{t} - V\right), \quad t\delta_1(x + Vt) = \delta_1\left(\frac{x}{t} + V\right)$$

the boundary conditions (2.13) can be written as

$$\operatorname{Im} z = 0, \quad \operatorname{Re} F\left(\frac{1}{z}\right) = -pV\,[\delta_1(z - V) + \delta_1(z + V)] \qquad (2.14)$$

The solution of Dirichlet problem (2.14) has the following form:

$$F\left(\frac{1}{z}\right) = -\frac{pVi}{\pi(z - V)} - \frac{pVi}{\pi(z + V)} \qquad (2.15)$$

i.e.

$$F(z) = -\frac{2pV^{-1}iz}{\pi(V^{-2} - z^2)}$$
$$W'(z) = \frac{2pV^{-1}c_2^{-2}i\sqrt{c_1^{-2} - z^2}}{\pi\mu(V^{-2} - z^2)S(z)}$$

The displacements and the stresses can be obtained from here with the aid of the formulas (1.30), (1.32) – (1.34) for $L = 1$.

3. Crack problems. We consider some problems of the motion of the cracks with constant velocities; these problems present interest in the fracture mechanics.

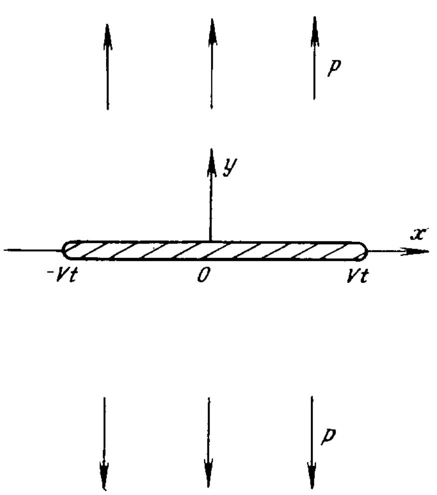

Fig. 1

Problem 3.1. Let us assume that at the initial instant $t = 0$ a cut appears at the origin of the coordinates and it starts to extend with a constant velocity V in both directions of the x-axis; the sides of the cut are subject to a constant normal load p (Fig. 1). We assume that $V < c_R$, where c_R is the velocity of the Raleigh surface waves.

The boundary conditions of the problem for the half-plane $y > 0$ have the form

$$\sigma_y = -p, \quad \tau_{xy} = 0 \qquad (3.1)$$
$$\text{for } y = 0, \quad |x| < Vt$$
$$v = 0, \quad \tau_{xy} = 0 \quad \text{for } y = 0, \quad |x| > Vt$$

In the unperturbed domain

$$\sigma_x = \sigma_y = \tau_{xy} = 0 \quad \text{for} \quad x^2 + y^2 \geqslant C_1{}^2 t^2 \tag{3.2}$$

Obviously, in this problem the stresses are homogeneous functions. According to formula (1.30) for $L = 1$, and also (1.35) and (1.36), the boundary conditions (3.1) lead us to the following boundary value problem:

$$\begin{aligned} &\operatorname{Im} z = 0, \ |\operatorname{Re} z| < V^{-1}, \quad \operatorname{Re} W(z) = 0 \\ &\operatorname{Im} z = 0, \ |\operatorname{Re} z| > V^{-1}, \quad \operatorname{Im} W'(z) = 0 \end{aligned} \tag{3.3}$$

To solve this problem it is necessary to know the behavior of the analytic function $W(z)$ for $|z| \to V^{-1}$ and for $z \to \infty$. The point at infinity of the plane z corresponds to the origin of the physical plane, where the displacement u is equal to zero and the displacement v is bounded. Taking into account the representations (1.30), (1.33) and (1.34), we have from here

$$z \to \infty, \quad \operatorname{Re} W(z) = O(1), \quad \operatorname{Im} W(z) = 0 \tag{3.4}$$

Integrating the second condition (3.3) and taking into account (3.4), we can write

$$\operatorname{Im} z = 0, \quad |\operatorname{Re} z| > V^{-1}, \quad \operatorname{Im} W(z) = 0 \tag{3.5}$$

The displacement v near the end of the crack $x = Vt$ becomes zero directly proportional [17] to the factor $(Vt - x)^{1/2}$; consequently, according to the formulas (1.30), (1.32) and (1.34), the function $W(z)$ can be written

$$W(z) = 0\,[\,(z \pm V^{-1})^{-1/2}] \quad \text{for} \quad z \to \mp V^{-1} \tag{3.6}$$

The solution of the boundary value problem (3.3) – (3.6) has the form [24, 25]

$$W(z) = \frac{Az + B}{\sqrt{z^2 - V^{-2}}} \tag{3.7}$$

$$\left(\sqrt{z^2 - V^{-2}} = z + O(z^{-1}), \ z \to \infty\right)$$

where A and B are some constants (the constant A is real). Because of the symmetry condition, the displacement v at the crack is an even function of x, therefore, from here, because of the formulas (1.35) and (3.7), we obtain $B = 0$. Consequently,

$$W(z) = \frac{Az}{\sqrt{z^2 - V^{-2}}}, \qquad W'(z) = -\frac{AV^{-2}}{(z^2 - V^{-2})^{3/2}} \tag{3.8}$$

With the aid of the formulas (1.35) and (1.36) we obtain from here the displacement at the crack and the stress σ_y at the continuation of the crack

$$v = AV^{-1}\sqrt{(Vt)^2 - x^2} \quad \text{for} \quad y = 0, \ |x| \leqslant Vt \tag{3.9}$$

$$\sigma_y = A\mu\left(\frac{c_2}{V}\right)^2 \operatorname{Re} \int_{c_1^{-1}}^{t/x} \frac{S(\tau)\,d\tau}{\sqrt{c_1^{-2} - \tau^2}\,(\tau^2 - V^{-2})^{3/2}} \quad \text{for} \quad y = 0, \ Vt < x < c_1 t. \tag{3.10}$$

The constant A is determined from the conditions (3.1), which, by virtue of (3.10), becomes

$$A\mu\left(\frac{c_2}{V}\right)^2 \operatorname{Re} \int_{c_1^{-1}}^{M} \frac{S(\tau)\,d\tau}{\sqrt{c_1^{-2} - \tau^2}\,(\tau^2 - V^{-2})^{3/2}} = -p \quad (V^{-1} < M \leqslant \infty) \tag{3.11}$$

The computation of the integral in (3.11), which exists in the sense of the principal value, leads us to the following expression:

$$A = \frac{pV^2}{\mu c_2^2 J}, \quad J = \frac{1}{V^{-1}(V^{-2} - c_1^{-2})} \{[c_2^{-4} + 4c_1^{-2}(V^{-2} - c_1^{-2})] K(\omega_1) -$$
$$[c_2^{-4} - 4V^{-2}(c_1^{-2} + c_2^{-2}) + 8V^{-4}] E(\omega_1) -$$
$$4c_2^{-2}(V^{-2} - c_1^{-2}) K(\omega_2) + 8V^{-2}(V^{-2} - c_1^{-2}) E(\omega_2)\}, \tag{3.12}$$
$$\omega_k = \sqrt{1 - \frac{c_k^{-2}}{V^{-2}}} \quad (k = 1, 2)$$

Here K and E are the complete elliptic integrals of the first and second kind, respectively. The computation of all the quantities which represent physical interest can be found in the Broberg's paper [6], whose solution is much more complex. If on the obtained solution we superpose a homogeneous extension $\sigma_y = p$, then, obviously, we obtain the solution of the problem of the propagation of a cut with free edges.

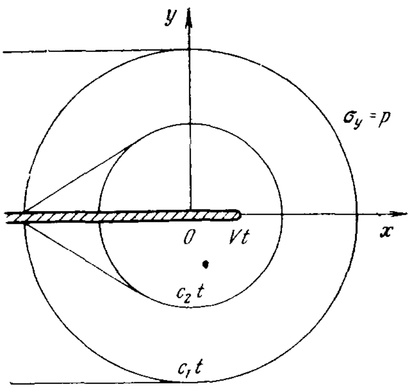

Fig. 2

We assume now that the load on the cut in Broberg's problem increases direct proportionally with time, i.e. instead of (3.1) we have the boundary condition $\sigma_y = -pt$, where $p = \text{const}$. Obviously, the solution of this problem can be obtained from the solution of Broberg's problem if in the latter the displacements and the stresses are replaced by the velocity and the derivatives of the stresses with respect to time.

Problem 3.2. Let us find the solution of Baker's problem [5]: the semi-infinite cut $x < 0$, $y = 0$ appears abruptly at time $t = 0$ and starts to expand along the x-axis with constant velocity V (Fig. 2).

The boundary conditions have the form

$$\sigma_y = 0, \ \tau_{xy} = 0 \quad \text{for} \quad y = 0, \ x < Vt \tag{3.13}$$
$$v = 0, \ \tau_{xy} = 0 \quad \text{for} \quad y = 0, \ x > Vt$$

With the aid of (1.35) – (1.39) the conditions (3.13) lead to the following Hilbert boundary value problem:

$$\operatorname{Im} z = 0, \quad \operatorname{Re} z < c_1^{-1}, \quad \operatorname{Re} z > V^{-1}, \quad \operatorname{Re} F(z) = 0 \tag{3.14}$$
$$\operatorname{Im} z = 0, \quad c_2^{-1} \leqslant \operatorname{Re} z \leqslant V^{-1}, \quad \operatorname{Im} F(z) = 0$$
$$\operatorname{Im} z = 0, \quad c_1^{-1} < \operatorname{Re} z \leqslant c_2^{-1}, \quad \operatorname{Im} \frac{F(z)}{S(z)} = 0$$

The solution of this problem, symmetric with respect to the real axis, with the asymptotic condition (3.6), has the form [24, 25]

$$F(z) = \frac{Ai(z - c_R^{-1})}{\sqrt{z - c_1^{-1}}\,(z - V^{-1})^{3/2}} \exp \Gamma(z)$$
$$\Gamma(z) = -\frac{1}{\pi} \int_{c_1^{-1}}^{c_2^{-1}} \operatorname{arctg} \frac{4\tau^2 \sqrt{(\tau^2 - c_2^{-2})(c_2^{-2} - \tau^2)}}{(c_2^{-2} - 2\tau^2)^2} \frac{d\tau}{\tau - z} \tag{3.15}$$

Here the radical $\sqrt{z - V^{-1}}$ is single-valued in the plane z with the cut $(-\infty, V^{-1})$ along the real axis, $\sqrt{z - V^{-1}} > 0$ for $\operatorname{Im} z = 0$, $\operatorname{Re} z > V^{-1}$, A is a real constant, determined from the condition for $x^2 + y^2 \geqslant c^2{}_1 t^2$ (Fig. 2) and is

$$A = \frac{p}{J}, \quad J = \operatorname{Re} \int\limits_M^{c_1^{-1}} \frac{(\tau - c_R^{-1}) \exp \Gamma^-(\tau)}{\tau \sqrt{\tau - c_1^{-1}} (V^{-1} - \tau)^{3/2}} d\tau \quad (V^{-1} < M \leqslant \infty) \tag{3.16}$$

where $\Gamma^-(z)$ is the limiting value of the function $\Gamma(z)$ from below. The investigation of the solution can be found in paper [5], where the cumbersome Wiener-Hopf technique has been applied for the determination of the solution.

Problem 3.3. We assume that a cut $-Vt \leqslant x \leqslant Vt$, $y = 0$ starts to extend under the action of instantaneous impulses of magnitude I, concentrated at the origin, directed along the y-axis and in the opposite directions symmetrically with respect to the x-axis. The edges of the cut are free of loads and we have zero initial conditions. The boundary conditions for the semi-plane $y > 0$ have the form

$$\begin{aligned} &\sigma_y = -I\delta_1(x)\,\delta_1(t), \quad \tau_{xy} = 0 \quad \text{for} \quad y = 0, \ |x| < Vt \\ &v = 0, \quad \tau_{xy} = 0 \quad \text{for} \quad y = 0, \ |x| > Vt \ \ (V < c_R) \end{aligned} \tag{3.17}$$

In this case the displacement potentials φ and ψ are homogeneous functions. Because of (1.35) and (1.36) we arrive at the following problem:

$$\begin{aligned} &\operatorname{Im} z = 0, \quad |\operatorname{Re} z| < V^{-1}, \quad \operatorname{Re} W'(z) = 0 \\ &\operatorname{Im} z = 0, \ |\operatorname{Re} z| > V^{-1}, \ \operatorname{Im}\left[\frac{zS(z)}{\sqrt{z^2 - c_1^{-2}}} W'(z)\right] = \frac{Ic_2^{-2}}{\mu}\delta_1\left(\frac{x}{t}\right). \\ &(z = t/x) \end{aligned} \tag{3.18}$$

This problem can be written in the form

$$\begin{aligned} &\operatorname{Im} z = 0, \ |\operatorname{Re} z| < V, \ \operatorname{Im} \frac{1}{z^2} W'\left(\frac{1}{z}\right) = -\frac{Ic_2^{-2}}{2\mu(c_2^{-2} - c_1^{-2})}\delta_1(z) \\ &\operatorname{Im} z = 0, \quad |\operatorname{Re} z| > V, \quad \operatorname{Re} \frac{1}{z^2} W'\left(\frac{1}{z}\right) = 0 \end{aligned} \tag{3.19}$$

since $S(z) = -2z^2(c_2^{-2} - c_1^{-2}) + O(1)$ for $z \to \infty$.

Due to the formulas (1.31) and (1.36), for $z \to V^{-1}$ the function $W(z)$ has order $O(\sqrt{z - V^{-1}})$, since the stress σ_y at the end of the crack has a singularity of the form $r^{-1/2}$.

The solution of the problem (3.19), under the indicated asymptotic condition, has the form

$$\frac{1}{z^2} W'\left(\frac{1}{z}\right) = \frac{Ic_2^{-2}\sqrt{z^{-2} - V^{-2}}}{2\pi\mu(c_2^{-2} - c_1^{-2})} \tag{3.20}$$

Hence

$$W'(z) = \frac{Ic_2^{-2}}{2\pi\mu(c_2^{-2} - c_1^{-2})} \frac{\sqrt{z^2 - V^{-2}}}{z^2} \tag{3.21}$$

The displacements and the stresses are determined with the aid of the representations (1.31) – (1.36). For example, the displacements of the edges of the cut are

$$v = \frac{Ic_2^{-2}V^{-1}}{2\pi\mu(c_2^{-2} - c_1^{-2})} \frac{\sqrt{(Vt)^2 - x^2}}{t} \quad (y = 0, \ |x| < Vt) \tag{3.22}$$

In fracture mechanics of fundamental importance is the stress field near the end of the crack, described by the stress intensity factor K_I

$$K_I = \lim_{x \to Vt} \left[\sigma_y \sqrt{2\pi (x - Vt)}\right] \tag{3.23}$$

In the case under consideration this factor is

$$K_I = -\frac{JS(V^{-1})}{2\pi^{1/2} V^{-1/2} (V^{-2} - c_1^{-2})^{1/2} (c_2^{-2} - c_1^{-2})\, t^{3/2}} \tag{3.24}$$

Problem 3.4. We give the solution of the problem, similar to the previous one, where we assume that the crack extends under the action of a force pt, increasing with time and concentrated at the origin.

The boundary conditions are

$$\begin{aligned} &\sigma_y = -pt\delta_1(x), \quad \tau_{xy} = 0 \quad \text{for} \quad y = 0, \ |x| < Vt \\ &v = 0, \quad \tau_{xy} = 0 \quad \text{for} \quad y = 0, \ |x| > Vt \ (V < c_R) \end{aligned} \tag{3.25}$$

Obviously, in this case the stresses are homogeneous functions. The solution of this boundary value problem is found as in the previous problem. It has the form

$$W'(z) = \frac{pc_2^{-2}}{2\pi\mu (c_2^{-2} - c_1^{-2})} \frac{z^2}{(z^2 - V^{-2})^{3/2}} \tag{3.26}$$

In this case, according to (1.30), (1.32) and (3.26), we obtain the displacement of the edges of the crack (for $y = 0$, $|x| < Vt$) and the stress intensity factor,

$$v = \frac{pc_2^{-2} t}{2\pi\mu (c_2^{-2} - c_1^{-2})} \left[\ln \left| \frac{Vt + \sqrt{(Vt)^2 - x^2}}{x} \right| - 2\, \frac{\sqrt{(Vt)^2 - x^2}}{Vt}\right] \tag{3.27}$$

$$K_I = -\frac{pS(V^{-1})\, t^{1/2}}{2\pi^{1/2} (V^{-2} - c_1^{-2})^{1/2} (c_2^{-2} - c_1^{-2})} \tag{3.28}$$

With the aid of the developed method it is easy to construct the solution for a moving crack with moving impulses or concentrated forces. Using these solutions as Green's functions, we can construct the analytic solution of the problem in the case of an arbitrarily loaded crack. The analysis of the obtained solution is not within the scope of the present paper.

4. Contact problems. Problem 4.1. We assume that a wedge-like punch with opening angle 2α is pressed against a semi-plane with constant velocity v_0 (Fig. 3). The wedge is assumed to be obtuse and symmetric with respect to the y-axis. At the initial instant $t = 0$ the vertex of the wedge coincided with the origin. There is no friction between the wedge and the elastic semi-plane. The initial conditions are zero. The boundary conditions of the problem are:

$$\begin{aligned} &v = v_0 t - \operatorname{ctg} \alpha \, |x|, \quad \tau_{xy} = 0 \quad \text{for} \quad y = 0, \ |x| < Vt \\ &\sigma_y = 0, \quad \tau_{xy} = 0 \quad \text{for} \quad y = 0, \ |x| > Vt \end{aligned} \tag{4.1}$$

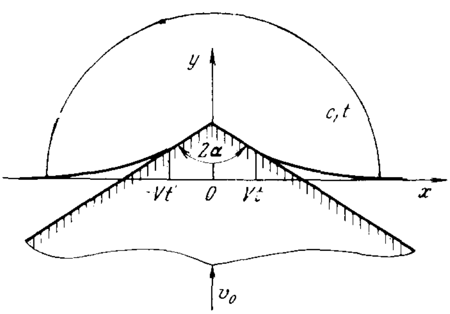

Fig. 3

We restrict ourselves to the case $V < c_R$. The quantity V is subject to determination from the solution.

In the problem under consideration the velocities are homogeneous functions, therefore we will make use of the representations (1.29), (1.32) − (1.39) for $L = \partial/\partial t$. The boundary conditions (4.1) lead us to the following boundary value problem:

$$\begin{aligned} &\operatorname{Im} z = 0, \qquad |\operatorname{Re} z| < V^{-1}, \qquad \operatorname{Re} F(z) = 0 \\ &\operatorname{Im} z = 0, \qquad |\operatorname{Re} z| > V^{-1}, \qquad \operatorname{Im} F(z) = 0 \end{aligned} \tag{4.2}$$

We seek the solution of the boundary value problem (4.2) which satisfies the following physical conditions: (1) the stress is bounded near the contact area $x = \pm Vt$, (2) at the angular point $x = 0$ the stresses have integrable singularities, (3) the stress $\sigma_y(x, 0, t)$ at the contact area is an even function of x. These conditions, as can be shown, allow us to construct a unique solution of the boundary value problem (4.2) in the following form (A is a real constant):

$$F(z) = -\frac{\mathrm{A}z}{\sqrt{z^2 - V^{-2}}} \tag{4.3}$$

The quantities A and V are obtained from the following conditions under the punch:

$$\frac{\partial v}{\partial t} = v_0, \ \frac{\partial v}{\partial x} = -\operatorname{ctg}\alpha \quad \text{for} \quad y = 0, \ 0 < x < Vt \tag{4.4}$$

These conditions by (1.29), (1.35) and (1.38) become

$$\frac{\mathrm{A}c_2^{-2}}{\mu} \mathrm{R} \int_{c_1^{-1}}^{V^{-1}} \frac{\varphi(\tau)\, d\tau}{S(\tau)} = v_0 \tag{4.5}$$

$$\frac{\mathrm{A}c_2^{-2}}{\mu} \operatorname{Re} \int_{c_1^{-1}}^{V^{-1}} \frac{\tau\varphi(\tau)\, d\tau}{S(\tau)} = \operatorname{ctg}\alpha, \quad \varphi(\tau) = -\frac{\sqrt{\tau^2 - c_1^{-2}}}{\sqrt{V^{-2} - \tau^2}}$$

From (4.5) we obtain the constant A and the equation for V

$$\begin{aligned} &\mathrm{A} = \frac{\mu v_0}{c_2^{-2} J_1}, \qquad \frac{J_1(V)}{J_2(V)} = v_0 \operatorname{tg}\alpha \\ &J_1 = \operatorname{Re} \int_{c_2^{-1}}^{V^{-1}} \frac{\varphi(\tau)\, d\tau}{S(\tau)} + \int_{c_1^{-1}}^{c_2^{-1}} \frac{(c_2^{-2} - 2\tau^2)^2 \varphi(\tau)\, d\tau}{S(\tau) S_1(\tau)} \\ &J_2 = \operatorname{Re} \int_{c_2^{-1}}^{V^{-1}} \frac{\tau\varphi(\tau)\, d\tau}{S(\tau)} + \int_{c_1^{-1}}^{c_2^{-1}} \frac{\tau(c_2^{-2} - 2\tau^2)^2 \varphi(\tau)\, d\tau}{S(\tau) S_1(\tau)} \\ &S_1(\tau) = (c_2^{-2} - 2\tau^2)^2 - 4\tau^2 \sqrt{c_1^{-2} - \tau^2} \sqrt{c_2^{-2} - \tau^2} \end{aligned} \tag{4.6}$$

With the aid of the formulas (4.3) and (1.39) for $L = \partial / \partial t$ we obtain the distribution of the pressure under the punch and the amount of the total force

$$\sigma_y = -\mathrm{A} \ln \left| \frac{Vt + \sqrt{(Vt)^2 - x^2}}{x} \right| \quad \text{for} \quad y = 0, \ |x| \leqslant Vt \tag{4.7}$$

$$P = -\int_{-Vt}^{Vt} \sigma_y(x, 0, t)\, dx = \pi \mathrm{A} V t \tag{4.8}$$

This problem has been studied in a much more complicated manner in [18] . However, the author has taken the value of the velocity V equal to $V = v_0 \operatorname{tg} \alpha$, while the value of V has to be found from the solution of the problem according to equation (4.6).

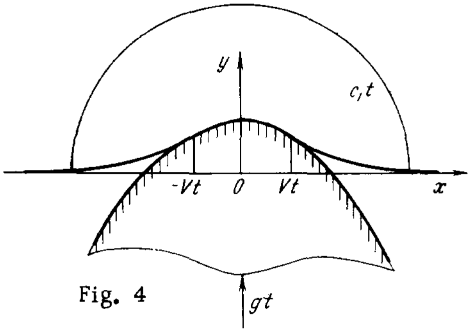

Fig. 4

Problem 4.2. We consider the uniformly accelerated indentation of a parabolic punch, symmetric with respect to the y-axis, into a semi-plane (Fig. 4). We assume zero initial conditions.

The boundary conditions of the problem are

$$\begin{aligned} &v = {}^1/_2 g t^2 - b x^2, \quad \tau_{xy} = 0 \quad \text{for} \quad y = 0, \ |x| < Vt \\ &\sigma_y = 0, \ \tau_{xy} = 0 \quad \text{for} \quad y = 0, \ |x| > Vt \end{aligned} \tag{4.9}$$

where b and g are specified constants. The quantity V has to be determined in the process of the solution. In this problem the accelerations are homogeneous functions, and so we apply the representations (1.29), (1.32) – (1.39) with $L = \partial^2 / \partial t^2$. The boundary conditions (4.9) lead us to the boundary value problem (4.2) for the function $F(z)$. We seek the solution of this problem which satisfies the following physical conditions: (1) the stresses are bounded near the ends of the contact area $x = \pm Vt$ and at the origin for $x = 0$; (2) the stress $\sigma_y(x, 0, t)$ at the contact area is an even function of x. These additional conditions allow us to construct a unique solution of the boundary value problem (4.2) in the following form:

$$F(z) = \frac{2AV^{-1}z}{(z^2 - V^{-2})^{3/2}} \tag{4.10}$$

Here A is a real constant. The quantities A and V are obtained from the conditions under the punch, which follow from (4.9)

$$\frac{\partial^2 v}{\partial t^2} = g, \ \frac{\partial^2 v}{\partial x^2} = -2b \quad \text{for} \quad y = 0, \ |x| < Vt \tag{4.11}$$

These conditions, with the aid of the formulas (4.10), (1.29), (1.35) and (1.38), can be reduced to the following relations:

$$\begin{gathered} A = \frac{\mu g}{c_2^{-2} J_1}, \qquad \frac{J_1(V)}{J_2(V)} = \frac{g}{2b} \\ J_1 = -\operatorname{Re} \int_{c_1^{-1}}^{M} \frac{\sqrt{\tau^2 - c_1^{-2}}\, d\tau}{S(\tau)(V^{-2} - \tau^2)^{3/2}}, \qquad J_2 = \operatorname{Re} \int_{c_1^{-1}}^{M} \frac{\tau^2 \sqrt{\tau^2 - c_1^{-2}}\, d\tau}{S(\tau)(V^{-2} - \tau^2)^{3/2}} \\ (V^{-1} < M \leqslant \infty) \end{gathered} \tag{4.12}$$

which serve for the determination of A and V (the second is an equation relative to V). From (4.10) and (1.39) we find the distribution of the stress under the punch and the amount of the total force acting on the punch

$$\sigma_y = -2A\sqrt{(Vt)^2 - x^2} \quad \text{for} \quad y = 0, \ |x| < Vt \tag{4.13}$$

$$P = -\int_{-Vt}^{Vt} \sigma_y(x, 0, t)\,dx = \pi A V^2 t^2 \qquad (4.14)$$

We note that on the basis of (4.7) and (4.13), the distribution of the pressure under the punch in the dynamical problems under consideration coincides with the statical ones, provided the width of the contact area is taken equal to $2Vt$.

Problem 4.3. We consider the dynamical problem of cleavage of a brittle body. We assume that at the initial instant $t = 0$ a semi-infinite wedge starts to move into an elastic plane with a constant velocity v_0, perpendicular to the x-axis in both directions (Fig. 5). Simultaneously, a rectilinear crack starts to extend from the end of the wedge into its continuation with constant velocity V the edges of the crack are free of stresses. The initial thickness of the wedge is taken to be zero; the remaining initial conditions are also zero. Obviously, the problem is symmetric with respect to the x-axis. Therefore it is sufficient to find the solution in the semi-plane $y > 0$. The boundary conditions of the problem have the form

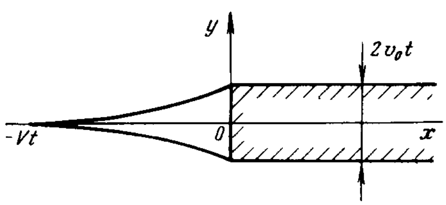

Fig. 5

$$\begin{aligned} &v = v_0 t, \quad \tau_{xy} = 0 \quad \text{for} \quad y = 0, \ x > 0 \\ &\sigma_y = 0, \quad \tau_{xy} = 0 \quad \text{for} \quad y = 0, \ -Vt < x < 0 \\ &v = 0, \quad \tau_{xy} = 0 \quad \text{for} \quad y = 0, \ x < -Vt \end{aligned} \qquad (4.15)$$

In this problem the velocities are homogeneous functions, therefore we will use the formulas (1.29) and (1.32) – (1.36) for $L = \partial / \partial t$.

The first boundary condition in (4.15) can be represented in the form

$$\frac{\partial^2 v}{\partial t^2} = v_0 \delta_1(t) \quad \text{for} \quad y = 0, \ x > 0 \qquad (4.16)$$

The boundary conditions (4.15), taking into account (4.16), lead us to the Keldysh-Sedov boundary value problem

$$\begin{aligned} &\operatorname{Im} z = 0, \quad -\infty < \operatorname{Re} z < -V^{-1}, \quad \operatorname{Im} W'(z) = 0 \\ &\operatorname{Im} z = 0, \quad -V^{-1} < \operatorname{Re} z < \infty, \quad \operatorname{Re} W'(z) = v_0\, \delta_1(z) \end{aligned} \qquad (4.17)$$

We seek the solution of this problem satisfying the following condition: the stresses have integrable singularities at the end of the crack for $x = -Vt$ and at the end of the wedge for $x = 0$. We can show that this additional condition allows us to construct the unique solution of the boundary value problem (4.17); this solution has the form [24, 25]

$$W'(z) = \frac{i v_0 V^{-3/2}}{\pi} \frac{Az - 1}{z\,(z + V^{-1})^{3/2}} \qquad (4.18)$$

where A is a real constant; the radical $\sqrt{z + V^{-1}} > 0$ at the lower side of the cut $(-V^{-1}, +\infty)$ of the real axis. We determine this constant from the condition

$$\sigma_y = 0 \quad \text{for} \quad y = 0, \ -Vt < x < 0$$

which takes the form

$$\operatorname{Re}\int_{-c_1^{-1}}^{M}\frac{i\,(A\tau-1)\,S(\tau)\,d\tau}{\tau\sqrt{c_1^{-2}-\tau^2}\,(\tau+V^{-1})^{3/2}}=0 \quad (-\infty\leqslant M<-V^{-1}) \tag{4.19}$$

From (4.19) we obtain

$$\begin{gathered}
A=\frac{J_1}{J_2},\ J_1=\int_{-c_1^{-1}}^{-c_2^{-1}}\frac{\varphi_2(\tau)\,d\tau}{\tau}+\int_{-c_2^{-1}}^{-V^{-1}}\frac{d}{d\tau}\left[\frac{\varphi_1(\tau)}{\tau}\right]\frac{d\tau}{\sqrt{\tau+V^{-1}}}-\varphi_0 \\
J_2=\int_{-c_1^{-1}}^{-c_2^{-1}}\varphi_2(\tau)\,d\tau+\int_{-c_2^{-1}}^{-V^{-1}}\frac{d}{d\tau}\left[\varphi_1(\tau)\right]\frac{d\tau}{\sqrt{\tau+V^{-1}}}+c_2^{-1}\varphi_0. \\
\varphi_0=\frac{2c_2^{-3}}{\sqrt{(c_2^{-2}-c_1^{-2})(V^{-1}-c_2^{-1})}},\ \varphi_1(\tau)=\frac{2S(\tau)}{\sqrt{\tau^2-c_1^{-2}}} \\
\varphi_2(\tau)=\frac{(c_2^{-2}-2\tau^2)^2}{\sqrt{\tau^2-c_1^{-2}}\,(\tau+V^{-1})^{3/2}}
\end{gathered} \tag{4.20}$$

With the aid of the formulas (1.30),(1.35),(1.36), taking into account (4.18) for $L=\partial/\partial t$ and (3.23), we find the displacement of the upper side of the crack and the stress intensity factor at the end of the dynamical crack

$$v=\frac{2v_0}{\pi}\left[\left(1+\frac{A}{V}\right)\sqrt{\frac{|x|\,(Vt+x)}{V}}+t\operatorname{arctg}\sqrt{\frac{Vt+x}{|x|}}\right] \tag{4.21}$$
$$(y=0,\ -Vt\leqslant x\leqslant 0)$$

$$K_I=-\frac{2\sqrt{2}\,\mu v_0\,(1-AV^{-1})\,S(V^{-1})\,t^{1/2}}{\pi^{1/2}c_2^{-2}V^{-1/2}\,(V^{-2}-c_1^{-2})^{1/2}} \tag{4.22}$$

Essentially through the efforts of Kolosov, Muskhelishvili, Westergaard, Galin and Radok, a class of statical and steady dynamic elasticity problems have been discovered, whose effective solution have been found with the methods of the theory of functions of a complex variable. The approach developed in this paper, based on the Smirnov-Sobolev functionally-invariant solutions, allows us to apply these methods to the effective solution of a similar class of dynamic problems of the theory of elasticity.

Appendix. Functionally-invariant solutions of the wave equation. We consider the wave equation in the plane case (a^{-1} is the wave propagation velocity)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=a^2\frac{\partial^2 u}{\partial t^2} \tag{A.1}$$

We seek solutions of the wave equation of the following form [1, 35]:

$$u(x,y,t)=f(\tau) \tag{A.2}$$

where $f(\tau)$ is an analytic function of τ, the variable τ being determined by the equation

$$l(\tau)\,t+m(\tau)\,x+n(\tau)\,y+p(\tau)=0 \tag{A.3}$$

Here $l(\tau)$, $m(\tau)$, $n(\tau)$, $p(\tau)$ are some analytic functions of the complex variable τ. Equation (A.3) determines τ as a function of the variables x, y and t. Computing the second order x, y and t derivatives of the function u with the aid of (A.3) and inserting them into Eq.(A.1), we obtain an equation of the following form (the prime denotes differentiation with respect to τ)

$$\frac{1}{\delta^1}\frac{\partial}{\partial\tau}\left[f'(\tau)\,\frac{m^2(\tau)+n^2(\tau)-a^2l^2(\tau)}{\delta'}\right]=0 \tag{A.4}$$

$$\delta' = l'(\tau)\,t + m'(\tau)\,x + n'(\tau)\,y + p'(\tau) \neq 0$$

It follows from (A.4) that $f(\tau)$ is the solution of the wave equation (A,1), provided the coefficients of the auxiliary equation (A.3) satisfy the relation

$$m^2(\tau)+n^2(\tau)-a^2\,l^2(\tau)=0 \tag{A.5}$$

Obviously, both the real and the imaginary part of the function $f(\tau)$ also satisfy the wave equation (A,1) [1]. The constructed solutions $f(\tau)$ of the wave equation have been found first by Smirnov and Sobolev in 1932 [1].

We consider the particular case of the functionally-invariant solutions of the wave equation when the function $p(\tau)=0$. Taking into account (A.5), we set

$$l(\tau)=1,\quad m(\tau)=-z,\quad n(\tau)=-\sqrt{a^2-z^2} \tag{A.6}$$

Then Eq. (A.3) takes the form

$$t-zx-\sqrt{a^2-z^2}\,y=0 \tag{A.7}$$

or

$$1-z\xi-\sqrt{a^2-z^2}\,\eta=0\quad(\xi=x/t,\ \eta=y/t) \tag{A.8}$$

Here the branch of the radical is fixed by the condition (z is the new complex variable)

$$\sqrt{a^2-z^2}=iz+O(z^{-1})$$

We can see from (A.7) that the solutions $f(z)$ of the wave equation are functions of the arguments ξ and η, i.e. homogeneous functions of x, y, t of zero dimension.

Let us investigate in detail Eq. (A.8). The radical $\sqrt{a^2-z^2}$ is single-valued in the plane of the complex variable z with the cut $(-a, +a)$ along the real axis. Solving Eq. (A.8) relative to z, we obtain

$$z=\frac{\xi-i\eta\sqrt{1-a^2(\xi^2+\eta^2)}}{\xi^2+\eta^2}=\frac{xt-iy\sqrt{t^2-a^2(x^2+y^2)}}{x^2+y^2} \tag{A.9}$$

Here the radicals have positive sign for

$$i\sqrt{t^2-a^2(x^2+y^2)}>0,\ x^2+y^2>a^{-2}t^2,\ y\to+0 \tag{A.10}$$

For fixed ξ and η by virtue of (A.8) we have a straight line. We will consider that part of the line on which $t>0$ and we will call the half-line a ray. According to (A.10), these rays form a conical bundle with vertex in the origin and with an opening angle $\operatorname{arctg} a^{-1}$ at the vertex, the axis of the bundle being the t-axis. Equation (A.8) or (A.9) makes to correspond to the rays of this bundle the complex values of the plane z with the cut $(-a, +a)$ along the real axis. We note that to the rays which form the surface of the bundle

$$\xi^2+\eta^2=a^{-2}\quad\text{or}\quad x^2+y^2=a^{-2}t^2 \tag{A.11}$$

there correspond the points of the cut in the z-plane. To the bundle axis $x=y=0$ or $\xi=\eta=0$ there corresponds the point at infinity of the plane. To the half-plane $y>0$ ($y<0$) there corresponds the half-plane $\operatorname{Im} z<0$ ($\operatorname{Im} z>0$). We investigate the values of z for those points (ξ,η), which lie outside the bundle mentioned above, i.e. for the points where we have

$$\xi^2+\eta^2\geqslant a^{-2}\quad\text{or}\quad x^2+y^2\geqslant a^{-2}t^2 \tag{A.12}$$

Equation (A. 8), under the condition (A. 12), gives two real roots belonging to the segment $(-a, +a)$

$$z_{1,2} = \frac{\xi \pm \eta \sqrt{a^2(\xi^2+\eta^2)-1}}{\xi^2+\eta^2} = \frac{xt \pm y \sqrt{a^2(x^2+y^2)-t^2}}{x^2+y^2} \qquad (A.13)$$

Let $M_0(x_0, y_0, t_0)$ be some point outside the conical bundle, its coordinates satisfying inequality (A. 12), and let z_{10} and z_{20} be the corresponding values of z according to (A. 13). Inserting the values $z = z_{10}$, $z = z_{20}$ into Eq. (A. 8), we have two planes in the space xyt which pass through the point M_0. Hence, to each value z_1 on the cut $(-a, +a)$, there corresponds some plane in the space xyt. This plane passes through the ray of the surface of the conical bundle corresponding to the value $z = z_1$ and is tangent to the surface. Otherwise the plane would intersect this surface and part of it would be inside the conical bundle. It would follow that to a point lying inside the bundle there correponds a real value $z = z_0$. According to (A. 9) and (A. 10) this is not possible.

Let $f(z)$ be an analytic single-valued function on the plane with the cut $(-a, +a)$ and assume that the corresponding solution

$$u(x, y, t) = \operatorname{Re} f(z) \qquad (A.14)$$

is defined inside the conical bundle (A. 10). We indicate a method of continuous extension of this solution into the space outside the conical bundle (A. 12). We consider the family of half-planes P_+, tangent to the surface of the bundle (A. 11) of one direction, taking in (A. 13), for example, only the upper plus sign. These tangent half-planes will not intersect and they fill out a part of the space outside the bundle. At one of these planes $f(z)$ maintains a constant value and we can define in a single-valued manner the solution $u(x, y, t)$ outside the bundle, making use of the same formula (A. 14) which gives the solution inside the conical bundle. In a similar manner we can extend the solution outside the conical space along the half-planes P_- of the other direction. We can decompose the function u into two terms $u = u_1(z) + u_2(z)$ and extend one of them along the half-tangents P_+, and the other one along the half-tangents P_-. It follows from here that the methods of extensions form an uncountable set.

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ON EQUILIBRIUM INSTABILITY IN HOLONOMIC MECHANICAL SYSTEMS WITH PARTIAL DISSIPATION

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We prove a theorem on the instability of the equilibrium of a dissipative system in the absence of a maximum of the force function. The dissipation is partial and is absent only in one of the degrees of freedom. The proof is based not on the linearization of the differential equations but on Liapunov's direct method and uses a somewhat modified form of Krasovskii's theorem. The instability is established for systems with arbitrary nonlinear dissipative forces and an isolated equilibrium.

1. Statement of the problem. Let $q' = (q_1, q_2, \ldots, q_n)$ be the generalized coordinates of a holonomic mechanical system with n degrees of freedom (here and later the prime denotes transposition). We assume that the kinetic energy is a quadratic form in the generalized velocities $q_1^{\cdot}, q_2^{\cdot}, \ldots, q_n^{\cdot}$

$$2T(q, q^{\cdot}) = \sum_{i,j=1}^{n} a_{ij}(q) q_i^{\cdot} q_j^{\cdot} = (q^{\cdot})' A(q) q^{\cdot} \tag{1.1}$$

where $A(q) = \| a_{ij} \|$. We assume that the functions $a_{ij}(q)$ are continuously differentiable in some neighborhood of the point $q = 0$, the matrix A is symmetric, and quadratic form (1.1) is positive definite in q.

Let the force function $U(q)$ also be continuously differentiable and, besides conservative forces, let there act only dissipative forces, so that the equations of motion have the form